On a Chen-Liu approach to the perturbed KdV equation

1986 J. Phys. A: Math. Gen. 19 L771

(http://iopscience.iop.org/0305-4470/19/13/003)

J. Phys. A: Math. Gen. 19 (1986) L771–L774. Printed in Great Britain

LETTER TO THE EDITOR

On a Chen–Liu approach to the perturbed KdV equation

K Roy Chowdhury and A Roy Chowdhury

High Energy Physics Division, Department of Physics, Jadavpur University, Calcutta-700 032, India

Received 21 January 1986

Abstract. We have shown that it is possible to deduce a Lax pair for the perturbed KdV equation by using the technique of Chen and Liu.

Recently there have been various attempts to extend the inverse scattering approach to a different class of non-linear problems (Bullough and Caudrey 1980). In this context the question of partial or complete integrability of some perturbed equations are of utmost importance, because the idealised equations never occur in actual physical situations. In a recent communication it has been shown by Kodama (1985) that the well known KdV equation, when perturbed by some additional terms of a particular nature, may be mapped via Birkoff's technique to an integrable system. Here we show that by following a simplified approach of Chen and Liu (1979) we can deduce a Lax pair for the perturbed KdV system correct up to first order in the perturbation parameter.

Let us consider the perturbed KdV equation written in the form

$$u_t = 6uu_x + u_{xxx} + \varepsilon K(u, u_x \ldots) \tag{1}$$

with $K(u)$ given as

$$K(u) = a_1 u_{5x} + a_2 u u_{3x} + a_3 u_x u_{2x} + a_4 u^2 u_x. \tag{2}$$

To apply the Chen–Liu approach we first of all linearise equation (1) by setting $u \to u + \eta\psi$ where η is a small parameter other than ε; whence we obtain the following equation for ψ:

$$\psi_t = \psi_{xxx} + 6\,(\partial/\partial x)(u\psi) + \varepsilon Q\psi \tag{3}$$

with

$$Q\psi = a_1\psi_{5x} + a_2(u\psi_{3x} + \psi u_{3x}) + a_3(u_x\psi_x)_x + a_4(u^2\psi)_x \tag{4}$$

equating first-order terms in η.

Now consider the adjoint of (4) which is

$$\frac{\partial\psi^a}{\partial t} = \left(\frac{\partial^3}{\partial x^3} + 6u\frac{\partial}{\partial x}\right)\psi^a + \varepsilon[(\partial/\partial x)(a_1 - a_3 u_{xx} + a_4 u^2) + a_2 u\,\partial^3/\partial x^3 - a_3 u_x\,\partial^2/\partial x^2]\psi^a \tag{5}$$

where ψ^a stands for the adjoint wavefunction. It is now customary to consider the right-hand side of (5) to be the temporal part of the Lax pair we are searching for.

Therefore we put

$$A = A^0 + \varepsilon A \tag{6}$$

along with

$$A^0 = \partial_x^3 + 6u\partial_x$$
$$A^1 = (\partial/\partial x)(a_1 - a_3 u_{xx} + a_4 u^2) + a_2 u\, \partial/\partial x - a_3 u_x\, \partial^2/\partial x^2. \tag{7}$$

The next part of our analysis considers the derivation of the conserved quantities associated with the perturbed system ( ) for which we set

$$\psi^{a} = \exp\left(Kx - \omega t + \int_{-\infty}^{x} T\, dx\right) \tag{8}$$

where

$$T = \sum_{n=0}^{\infty} k^{-n} T_n$$

in equation (5). Equating powers of k^{-n} we arrive at the following recurrence relation for the T_i:

$$\begin{aligned} T_{n,t} = [(&T_{n-i-j} T_i T_j + 3T_{n+2} + 3T_{n-i+1} T_i) \\ &+ 3T_{n+1,x} + 3T_{n-i} T_{ix} + T_{n,xx}](1 + \varepsilon u a_2) \\ &+ [6u + \varepsilon(a_1 - a_3 u_{xx} + a_4 u^2)](T_{n-1} + T_n) \\ &- [2T_{n-1} + T_{n-i} T_i + T_{n,x}]\varepsilon a_3 u_x. \end{aligned} \tag{9}$$

Some of the T_i can be obtained from (9); we report a few of them:

$$\begin{aligned} T_1 &= -2u + \tfrac{1}{3}\varepsilon(-1 + 20u_{xx} + 30u^2) \\ T_2 &= -T_{1x} \\ T_3 &= -2u_{xx} + 2u^2 + \tfrac{1}{3}\varepsilon(18u_{4x} + 40uu_{2x} - 40u^3 - 40uu_x + 4u) \\ T_4 &= 2u_{3x} + 8uu_x + \tfrac{1}{3}\varepsilon(-16u_{5x} - 140uu_{3x} - 120u_x u_{2x} - 280u^2 u_x - 80uu_x) \\ T_5 &= -2u_{4x} - 12uu_{2x} - 10u_x^2 - 4u^3 + \tfrac{1}{3}\varepsilon(14u_{6x} + 108uu_{4x} + 388u_x u_{3x} - \tfrac{290}{3}u^4 \\ &\quad + \tfrac{298}{3}u_{2x}^2 + 80u^2 u_{2x} + \tfrac{388}{3}uu_x^2 + \tfrac{16}{3}u^2 + 2u_x + \tfrac{1}{3}u_{2x} + 80uu_{2x}). \end{aligned} \tag{10}$$

From these T_i we then construct ψ_i defined as

$$\psi_n = \frac{\delta}{\delta q} T_n = \sum_{i=\partial}^{\infty} (-1)^i \frac{\partial^i}{\partial x^i} \frac{\partial}{\partial q_{ix}} T_n \tag{11}$$

where q_{ix} denotes the ith derivative of q.

The interesting phenomenon that now takes place is that even in the case of the perturbed KdV equation the even ψ_i, i.e. ψ_2, $\psi_4, \ldots$, etc, all turn out to be zero. Explicitly we obtain

$$\begin{aligned} \psi_1 &= -2 + \tfrac{1}{3}\varepsilon(60u) \\ \psi_2 &= 0 \\ \psi_3 &= -4u + \tfrac{1}{3}\varepsilon(160u_{2x} + 240u^2) \\ \psi_4 &= 0 \\ \psi_5 &= -4(3u^2 + 2u_{2x}) + \tfrac{1}{3}\varepsilon(160u_{2x} + \tfrac{92}{3}u_x^2 - 360u_{4x} - \tfrac{1160}{3}u^3 + \tfrac{184}{3}uu_{2x} + \tfrac{32}{3}u). \end{aligned} \tag{12}$$

When ε becomes zero these ψ become those of Iino and Ichikawa (1982). These forms suggest that we can construct a recursion operator L of the form

$$L = \partial^2/\partial x^2 + a + \alpha\varepsilon(\partial^2/\partial x^2 + b) \tag{13}$$

with α being a numerical factor.

Now Lax representation demands that the perturbed KdV equation (1) should be a consequence of the linear pair

$$\psi_t = A\psi \qquad \text{and} \qquad L\psi = \lambda\psi \tag{14}$$

which is known to yield

$$L_t = [A, L]. \tag{15}$$

Now we rewrite L and A as

$$\begin{aligned} A &= A_0 + \varepsilon A_1 \\ L &= L_0 + \varepsilon L_1 \end{aligned} \tag{16}$$

where the subscripts '0' and '1' refer to the unperturbed or perturbed part of A and L. Further decomposition of A and L is possible in the sense of Iino and Ichikawa, by segregating the parts A_i or L_i depending on the fields or not. Thus we obtain

$$\begin{aligned} A &= A_0 + \varepsilon A_1 \\ &= A_0^0 + A_0^1 + \varepsilon(A_1^0 + A_1^1) \end{aligned}$$

and

$$\begin{aligned} L &= L_0 + \varepsilon L_1 \\ &= L_0^0 + L_0^1 + \varepsilon(L_1^0 + L_1^1) \end{aligned} \tag{17}$$

where the superscripts '0' and '1' refer to the zeroth or first order with respect to the dependence on the non-linear field variables.

Now from (15) we obtain

$$(\partial/\partial t)(L_0 + \varepsilon L_1) = [A_0 + \varepsilon A_1, L_0 + \varepsilon L_1]$$

or

$$\begin{aligned} \partial L_0/\partial t &= [A_0, L_0] \\ \partial L_1/\partial t &= [A_0, L_1] + [A_1, L_0] \end{aligned} \tag{18}$$

for first-order terms in ε. Now again from equation (15) we obtain

$$(\partial/\partial t)(L_0^0 + L_0^1) = [A_0^0 + A_0^1, L_0^0 + L_0^1].$$

Separating first- and zeroth-order terms

$$\begin{aligned} \partial L_0^0/\partial t &= [A_0^0, L_0^0] \\ \partial L_0^1/\partial t &= [A_0^0, L_0^1] + [A_0^1, L_0^0] \end{aligned} \tag{19}$$

which are the equations of Iino and Ichikawa which determine a. If we apply the same procedure to equation (18) we obtain

$$\partial L_1^1/\partial t = [A_0^0, L_1^1] + [A_0^1, L_1^0] + [A_1^0, L_0^1] + [A_1^1, L_0^0]. \tag{20}$$

The zeroth-order term of the perturbed part of L (i.e. α) is easily computed by comparing the highest derivative terms in (13) which yields

$$L_1^0 = \tfrac{8}{3}\,\partial^2/\partial x^2 \tag{21}$$

so that α in equation (13) is assigned the value $\frac{8}{3}$. Now we have to recourse to the linear mode coupling scheme to determine b, from equation (20), which yields

$$\begin{aligned}(b_t - b_{3x})\psi - (3b_{2x}\psi_x + 3b_x\psi_{2x}) \\ = [a_x + 20u_{5x} - 180u_xu_{2x} - 60uu_{3x}]\psi \\ + [-16u_{2x} + 40u_{4x} - 120u_x^2 - 120uu_{2x}]\psi_x \\ + [-32u_x + 20u_{3x}]\psi_{2x} + 30u_{2x}\psi_{3x} - 20u_x\psi_{4x}.\end{aligned} \tag{22}$$

Now for the linear mode coupling scheme we have set

$$\begin{aligned} u &= \sum u_k \exp[\mathrm{i}(Kx + K^3t)] \\ \psi &= \sum \psi_l \exp[\mathrm{i}(lx + l^3t)] \\ b &= \sum b_p \exp[\mathrm{i}(px + p^3t)]. \end{aligned} \tag{23}$$

Then equation (20) can be solved for b in operator form and we obtain

$$\begin{aligned} b = \tfrac{1}{18}\int (D^{-1}u) + \tfrac{5}{9}\int u\partial_x^3 + \tfrac{5}{6}\int u_x\partial_x^2 + \int (-\tfrac{8}{9}u + \tfrac{5}{9}u_{xx})\partial_x \\ + \int (-\tfrac{4}{9}u_x + \tfrac{10}{9}u_{3x}) - \tfrac{10}{3}(u_x + u)\int u_x + \tfrac{5}{9}\int u_{4x}D^{-1} \\ - \tfrac{5}{3}\left(u_{xx}\int uD^{-1} + u_{xxx}\int (D^{-1}u)D^{-1}\right). \end{aligned} \tag{24}$$

So b is completely determined in terms of the non-linear field and hence the Lax operator for the perturbed KdV equation is explicitly realised. It is important to note that in our above computation we have fixed the values of the constant a_i as in Kodama (1985).

References

Bullough R K and Caudrey P J (ed) 1980 *Solitons* (Berlin: Springer)
Chen H and Liu C S 1979 *Phys. Scr.* **20** 490
Iino K and Ichikawa Y 1982 *J. Phys. Soc. Japan* **51** 4091
Kodama Y 1985 *Physica* **16D** 14